Flow and critical velocity of an imbalanced Fermi gas through an optical potential

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Abstract

Optical lattices offer the possibility to investigate the superfluid properties of both Bose condensates and Fermionic superfluid gases. When a population imbalance is present in a Fermi mixture, this leads to frustration of the pairing, and the superfluid properties will be affected. In this contribution, the influence of imbalance on the flow of a Fermi superfluid through an optical lattice is investigated. The flow through the lattice is analysed by taking into account coupling between neighbouring layers of the optical lattice up to second order in the interlayer tunneling amplitude for single atoms. The critical velocity of flow through the lattice is shown to decrease monotonically to zero as the imbalance is increased to 100%. Closed-form analytical expressions are given for the tunneling contribution to the action and for the critical velocity as a function of the binding energy of pairs in the (quasi) two-dimensional Fermi superfluid and as a function of the imbalance.

I. INTRODUCTION

In the past several years, ultracold Fermi gases have become a major focus of research on the interplay between Cooper pairing, strong interactions, and reduced dimensionality. The appeal of this system stems from the fact that both the interaction strength and the geometry of the confinement potential can be controlled very precisely. Feshbach resonances can be applied to tune the s-wave scattering length over a wide range of positive and negative values, which allows to investigate the crossover between a regime of Bose-Einstein condensed (BEC) molecules[1] and a Bardeen-Cooper-Schrieffer (BCS) superfluid[2]. Optical lattices are used to reduce the dimensionality of the system to 1D or 2D or to create a 3D optical lattice[3, 4, 5]. Moreover, since the amount of atoms of each species in a Fermi mixture can also be controlled accurately, it has become possible to investigate the effect, on pair formation, of a population imbalance between the pairing partners[6, 7].

In this contribution, we investigate the case of an imbalanced Fermi superfluid in a one-dimensional optical potential. In previous work, it was shown that the gap and number equations can be solved exactly for an imbalanced two-dimensional Fermi gas[13] – this corresponds to a single layer in the optical potential. Here, we will take into account the coupling between the different layers in the 1D lattice, and derive expressions for the tunneling amplitude and the critical velocity of the imbalanced Fermi superfluid through the optical lattice. Tunneling of a balanced Fermi superfluid through a 1D optical lattice has been studied experimentally[5], as has the imbalanced 3D Fermi superfluid[6, 7]. However, to the best of our knowledge, tunneling of a imbalanced superfluid has not yet been investigated.

II. OPTICAL LATTICE

The system that we study in this contribution consists of a Fermi mixture with unequal amounts of two components, referred to as "spin up" and "spin down". This is loaded into a one-dimensional optical lattice (in the z-direction). The optical lattice is created by two counterpropagating laser beams with wave length λ and is modelled by a potential energy $V_0 \sin^2(2\pi z/\lambda)$. Under the influence of this lattice, the gas forms a stack of typically a few hundred quasi-2D layers containing several thousands of atoms each. The confinement

influences the way that interactions are taken into account. The interaction between atoms within one quasi-2D layer can be modelled by a 2D-pseudopotential $V = g\delta(\mathbf{r})$ where the strength g depends on the energy of the scattering atoms through

$$\frac{1}{g} = \frac{m}{4\hbar^2}\left[i - \frac{\ln(E/E_b)}{\pi}\right] - \int \frac{d^2\mathbf{k}}{(2\pi)^3}\frac{1}{(\hbar k)^2/m - E + i\varepsilon}. \tag{1}$$

Here m is the mass of the atoms, and E_b is the energy of the bound state that always exists in two dimensions[8], given by $E_b = (C\hbar\omega_L/\pi)\exp(\sqrt{2\pi}\ell_L/a_s)$, with a_s the (3D) s-wave scattering length of the fermionic atoms, $\omega_L = \sqrt{8\pi^2 V_0/(m\lambda^2)}$ and $\ell_L = \sqrt{\hbar/(m\omega_L)}$ and $C \approx 0.915$ (cf. Ref. [9]).

The goal of this contribution is to take into account the coupling between adjacent valleys of the optical potential. This coupling is due to tunneling of individual atoms from layer to layer and is characterised by the tunneling amplitude[10]

$$t = \frac{m\omega_L^2\lambda^2}{8\pi^2}\left[\frac{\pi^2}{4} - 1\right]e^{-(\lambda/4\ell_L)^2}. \tag{2}$$

III. PATH-INTEGRAL APPROACH

We follow the path-integral approach, as applied by Iskin and Sa de Melo [11] for an imbalanced gas, and as applied in Ref. [12] to the optical lattice. The action functional S for the fermionic mixture in the optical lattice can be written as a path integral over the exponential of an action functional, consisting of the contributions S_j of the *individual layers* and the contributions $S_{j,j+1}$ of *tunneling* between adjacent layers. That is, we write for the action $S = \sum_j (S_j + S_{j,j+1})$, with the single-layer action

$$S_j = \sum_{k,\sigma}\bar{\psi}^{(j)}_{k,\sigma}\left(-i\omega_n + \mathbf{k}^2 - \mu^{(j)}_\sigma\right)\psi^{(j)}_{k,\sigma} + g\sum_k \bar{\psi}^{(j)}_{k,\uparrow}\bar{\psi}^{(j)}_{-k,\downarrow}\psi^{(j)}_{-k,\downarrow}\psi^{(j)}_{k,\uparrow}, \tag{3}$$

and the tunneling action

$$S_{j,j+1} = \sum_{k,\sigma} t\left(\bar{\psi}^{(j+1)}_{k,\sigma}\psi^{(j)}_{k,\sigma} + \bar{\psi}^{(j)}_{k,\sigma}\psi^{(j+1)}_{k,\sigma}\right). \tag{4}$$

Here, we use $k = \{\mathbf{k}, \omega_n\}$ for the the 2D wave number $\mathbf{k}$ and the Matsubara frequency ω_n, and σ for the spin, so that $\bar{\psi}^{(j)}_{k,\sigma}$ and $\psi^{(j)}_{k,\sigma}$ represent the Grassmann variables for the fermionic fields in layer j. The system parameters are the interaction strength g, the tunneling amplitude t, and the chemical potentials $\mu^{(j)}_\sigma$ that fix the amounts of spin-up and spin-down particles. We use units $\hbar = 2m = 1$.

The partition sum is the functional integral of $\exp\{-S\}$ over the Grassmann variables. The interaction part is decoupled by introducing the Hubbard-Stratonovic fields Δ_j,$\bar{\Delta}_j$ in each layer, after which the functional integral over the Grassmann variables can be taken. Writing those fields as a function of amplitude and phase, $\Delta_j = |\Delta_j|\, e^{i\theta_j}$, we find the following effective action

$$S_{eff} = -\frac{1}{g}\sum_j |\Delta_j|^2 - \mathrm{Tr}\left\{\log\left[-\mathcal{G}^{-1}_{j,j'} + \mathcal{T}_{j,j'}\right]\right\} \tag{5}$$

Here, the trace is taken over k, over the spin variables and over the layer index. The single layer inverse Green's function is

$$-\,\mathcal{G}^{-1}_j = \delta_{j',j} \times \begin{pmatrix} -i\omega_n + \mathbf{k}^2 - \mu_{\uparrow,j} & |\Delta_j| \\ |\Delta_j| & -i\omega_n - \mathbf{k}^2 + \mu_{\downarrow,j} \end{pmatrix} \tag{6}$$

and the tunneling propagator is

$$\mathcal{T}_{j,j'} = [\delta_{j',j+1} + \delta_{j',j-1}] \times \begin{pmatrix} te^{-i(\theta_j - \theta_{j'})/2} & 0 \\ 0 & -te^{i(\theta_j - \theta_{j'})/2} \end{pmatrix} \tag{7}$$

Setting $t = 0$ we obtain the result for the imbalanced 2D Fermi superfluid, discussed in Ref. [13]. We write this result as $S_{eff}(t=0) = \sum_j \beta\Omega_j$ where $\beta = 1/k_B T$ is the inverse temperature, and Ω_j is the BCS free energy for layer j. If, on the other hand, we keep $t > 0$ but set $\mu_{\downarrow,j} = \mu_{\uparrow,j}$ we obtain the results for the balanced Fermi gas in an optical potential, derived in Ref. [12]. Here, we keep $t > 0$ and investigate $\mu_{\downarrow,j} \neq \mu_{\uparrow,j}$. We now treat the additional tunneling terms as a perturbation, and expand up to order t^2. Diagrammatically, the process we include in the self-energy of layer j consists of a particle tunneling back and forth over layer $j+1$ or $j-1$. The result is

$$S_{eff} = \sum_j \left[\beta\Omega_j + \sum_k \frac{2t^2\cos(\theta_{j+1}-\theta_j)\,|\Delta_j|\,|\Delta_{j+1}|}{[(\zeta_j - i\omega_n)^2 - (E^{(j)}_{\mathbf{k}})^2][(\zeta_{j+1} - i\omega_n)^2 - (E^{(j+1)}_{\mathbf{k}})^2]}\right] \tag{8}$$

where $\sum_k$ represents both the sum over Matsubara frequencies and the integral over $\mathbf{k}$. Here, $\mu_j = (\mu_{\uparrow,j} + \mu_{\downarrow,j})/2$ is the average chemical potential and $\zeta_j = (\mu_{\uparrow,j} - \mu_{\downarrow,j})/2$ is the difference in chemical potentials, and

$$E^{(j)}_{\mathbf{k}} = \sqrt{(k^2 - \mu_j)^2 + |\Delta_j|^2}, \tag{9}$$

$$\xi^{(j)}_{\mathbf{k}} = k^2 - \mu_j. \tag{10}$$

Note that up to this order of perturbation, the normal particles do not contribute to tunneling: when we interleave layers of superfluid with layers of normal gas (for example the excess spin population), the product $|\Delta_j|\,|\Delta_{j+1}|$ vanishes. When we assume that no such interleaving takes place and that moreover $|\Delta_j| \approx |\Delta_{j+1}|$, the integration over $\mathbf{k}$ and the Matsubara summation can be performed analytically and results in

$$S_{eff} = \sum_j \beta \left[\Omega_j + T_{j+1,j}\cos(\theta_{j+1} - \theta_j)\right], \tag{11}$$

with

$$T_{j+1,j} = \frac{t^2}{8\pi}\left(1 + \frac{\mu_j}{\sqrt{\mu_j^2 + |\Delta_j|^2}} - 2\frac{\sqrt{\zeta_j^2 - |\Delta_j|^2}}{\zeta_j}\right). \tag{12}$$

where for $\zeta_j < |\Delta_j|$ the last term is not present; this case corresponds to the known result for the balanced gas[12].

IV. JOSEPHSON REGIME AND CRITICAL VELOCITY

We will investigate the specific case where both the 2D density $n_\uparrow + n_\downarrow$ and the imbalance do not change significantly over the layers. In that case $T_{j+1,j}$, expression (12), is also not changing significantly from layer to layer, and we can drop the index j for $|\Delta|, \mu$ and ζ. The superfluid motion over the layers is then due to the phase differences over the layers. Using the results from [13], we can re-express $T_{j+1,j}$ as a function of the imbalance $\delta n/n = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow)$, and of the binding energy per particle $E_b/(2E_F)$, in units of the Fermi energy $E_F = (\hbar k_F)^2/(2m)$ with $k_F^2 = 2\pi(n_\uparrow + n_\downarrow)$.

Three regimes can be identified. The first regime occurs for large imbalance, $\delta n/n > E_b/(2E_F)$. Then no superfluidity occurs – the Fermi mixture is too imbalanced to support pairing. The second regime is the 'weak pairing' regime, characterised by $\delta n/n < E_b/(2E_F)$. In the three-dimensional case, one would refer to this as the BCS regime. Then we have, from Ref. [13], the following analytical results for the 2D case:

$$|\Delta|^2 = 2E_b\left[1 - \sqrt{2/E_b}(\delta n/n)\right] \tag{13}$$

$$\mu = 1 - (E_b/2)\left[1 - \sqrt{2/E_b}(\delta n/n)\right] \tag{14}$$

$$\zeta = \sqrt{|\Delta|^2 + (\delta n/n)^2} \tag{15}$$

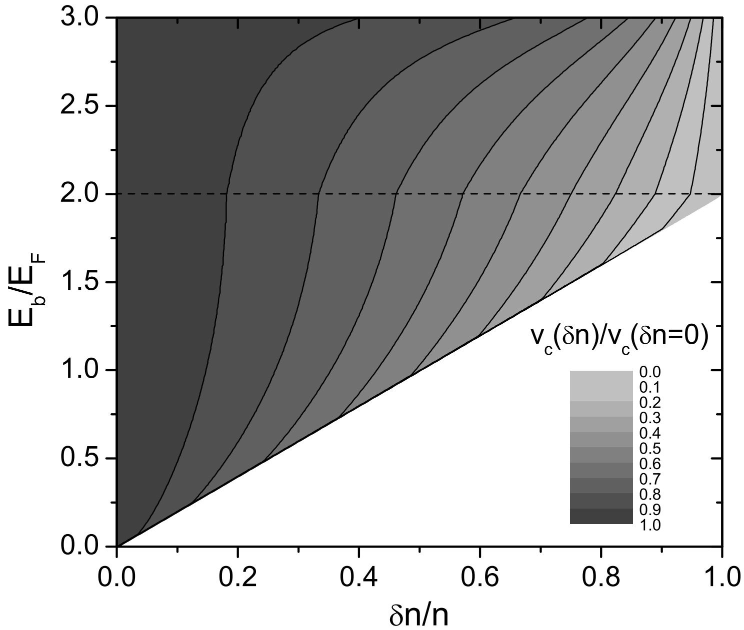

FIG. 1: Imbalance between the spin components reduces the critical velocity of a Fermi superfluid in an optical lattice. Here, the critical velocity relative to the critical velocity for a balanced gas is shown as a function of the pair binding energy E_b and the imbalance δn. The dashed line at $E_b = 2E_F$ separates the regime of weak pairing from that of strong pairing. For $E_b/E_F < 2\delta n/n$ (white region) superfluidity is suppressed.

This results in

$$T_{j+1,j} = \frac{t^2}{8\pi}\left[\frac{4E_F}{2E_F + E_b - \sqrt{2E_F E_b}\frac{\delta n}{n}} - \frac{2\sqrt{E_F}\frac{\delta n}{n}}{\sqrt{2E_b - 2\sqrt{2E_F E_b}\frac{\delta n}{n} + E_F\left(\frac{\delta n}{n}\right)^2}}\right]. \tag{16}$$

The third regime is that of strong pairing, characterised by $E_b/(2E_F) > 1$. In three dimensions this regime might be compared to the BEC regime. For the strong pairing regime, we find [13]

$$|\Delta|^2 = 2E_b\left(1 - \delta n/n\right) \tag{17}$$

$$\mu = 1 - \left(E_b/2\right)\left(1 - \delta n/n\right) \tag{18}$$

$$\zeta = \sqrt{|\Delta|^2 + \left[2(\delta n/n) - \mu\right]^2} \tag{19}$$

which results in

$$T_{j+1,j} = \frac{t^2}{8\pi}\left[\frac{4E_F}{2E_F + E_b - \sqrt{2E_F E_b}\frac{\delta n}{n}} \right.$$
$$\left. +2\frac{\left(2\frac{\delta n}{n} - 1\right) + \frac{E_b}{2}\left(1 - \frac{\delta n}{n}\right)}{\sqrt{\left(2\frac{\delta n}{n} - 1\right)^2 + \left(2\frac{\delta n}{n} + 1\right) E_b \left(1 - \frac{\delta n}{n}\right) + \frac{E_b^2}{4}\left(1 - \frac{\delta n}{n}\right)^2}}\right] \tag{20}$$

The presence of imbalance reduces $T_{j+1,j}$ from its result without imbalance. Since the critical velocity can be written as $v_c = \lambda T_{j+1,j}/(\hbar E_F)$ (see Ref. [14]), we find that that the imbalance also reduces the critical velocity for flow through the lattice. In Fig. 1, we show this reduction $v_c(\delta n)/v_c(\delta n = 0)$, as a function of E_b and $\delta n/n$. Note that for $E_b/E_F < 2\delta n/n$ superfluidity is not present. The two regimes of weak pairing and strong pairing can be distinguished by a kink in the contour lines.

V. CONCLUSIONS

The theory of flow through the lattice set up Ref. [12] is largely unmodified by the presence of imbalance. The main effect is the reduction of the tunneling coefficient $T_{j,j+1}$, expression (12). In this work, we kept δn constant and assume that $|\Delta|, \mu, \zeta$ vary slowly from one lattice site to another. Under these assumptions, we have derived closed analytical formulae for the tunneling contributions (and thus the critical velocity) for flow of an imbalanced superfluid through an optical lattice.

The assumption that $|\Delta|, \mu, \zeta$ vary slowly is good unless we have interleaving of normal gas and superfluid gas layers. If this is the case tunneling is suppressed because we take the optical potential to be deep enough such that the normal gas is pinned and flow is only due to phase coherence of the pair condensate. The assumption of constant δn is related; it also relies on the fact that there is no phase separation. This will be much more difficult to satisfy in practice, as coexistence may only be achievable at finite temperatures. In practice, at low temperature, there will be phase separation in the layer and the tunneling will only take place in the region of superfluid phase, and will be suppressed in the normal layer around it. This dynamical interplay between pinned normal state, and a phase separated superfluid are outside the scope of this paper, which relies on the possibility to create (albeit dynamically) an imbalanced superfluid.

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